Principal Series Part I.

Notation: F p-adic field $\mathcal{O}_F$, $\mathfrak{p} = (\varpi)$ $G = GL_2(F)$.

Recall: $(\rho_{\mu_1,\mu_2}, \mathcal{B}_{\mu_1,\mu_2}) = \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$ normalised ind. (smooth)

$$= \left\{ \varphi : G \to \mathbb{C} \text{ sm} \;\middle|\; \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \mu_1(a)\mu_2(d)\left|\frac{a}{d}\right|^{\frac{1}{2}}\varphi(g) \right\}$$

Want to know reducibility, irred. submods.

+ supercuspidals $\rightsquigarrow$ all irred. adm repn. of G.

Plan: $\mathcal{B}_{\mu_1,\mu_2} \longrightarrow \mathcal{K}_{\mu_1,\mu_2} \subset C^\infty(F^\times)$ ("Kirillov model") (This map may not be inj.)

$\xi \in \mathcal{K}_{\mu_1,\mu_2}$: $\rho\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\xi(x) = \tau_F(bx)\,\xi(ax)$

closure ←

closed cell open cell

$G = B \sqcup Bw^{-1}N$ Bruhat decomp. $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

$\varphi \in \mathcal{B}_{\mu_1,\mu_2}$ is determined by its values on ~~B~~wN

Set $\Phi_\varphi(x) = \varphi\left(w^{-1}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right)$.

For $|x|$ large Iwasawa decomp.

$$\Phi_\varphi(x) = \varphi\left(\underset{N}{\begin{pmatrix} 1 & -x^{-1} \\ & 1 \end{pmatrix}}\underset{M}{\begin{pmatrix} x^{-1} & \\ & x \end{pmatrix}}\underset{K}{\begin{pmatrix} 1 & \\ -x^{-1} & 1 \end{pmatrix}}\right) = \mu^{-1}(x)\,|x|^{-1}\varphi(e).$$

Set $\mu = \mu_1\mu_2^{-1}$.

$\Rightarrow$ $\mu(x)\,|x|\,\Phi_\varphi$ is const for $|x|$ large.

$$\mathcal{H}_\mu := \left\{ \Phi : F \to \mathbb{C} \text{ sm} \;\middle|\; \mu(x)|x|\Phi(x) \text{ const for } |x| \text{ large} \right\}$$

$\mathcal{B}_{\mu_1,\mu_2} \overset{1-1}{\longleftrightarrow} F_\mu \longrightarrow \hat{F}_\mu$ ↶ $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ acts by $\tau_F(b\cdot)$

↶ $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ acts by translation

$\Phi \overset{\text{Fourier transf.}}{\longmapsto} \boxed{\int_F \Phi(y)\overline{\tau_F(xy)}\,dy} \cdot \mu_2(x)|x|^{\frac{1}{2}}.$ — not well-defined

+ some adjustment to make $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ act in the correct way.

$\Phi \longmapsto \hat{\Phi}(x)\,\mu_2(x)|x|^{\frac{1}{2}}$

Fourier transf. $\tilde{\Phi}$ makes sense as distribution.

$$\langle \tilde{\Phi}, \underset{\mathcal{S}(F)}{f} \rangle = \langle \Phi, \hat{f} \rangle$$

Furthermore if f comes from $\mathcal{S}(F^\times)$, then $\tilde{\Phi}$ is "represented" by $\hat{\Phi}$ ← a fun. on $F^\times$.

$$\hat{\Phi}(x) := \sum_{n\in\mathbb{Z}} \int_{\varpi^n \mathcal{O}_F^\times} \Phi(y)\overline{\tau_F(xy)}\,dy. \qquad x \in F^\times.$$

Lemma 9

① The series $\hat{\Phi}$ unif. conv. on cpt of $F^\times$

② $\Phi \mapsto \hat{\Phi}$ is inj. unless $\mu = |\ |^{-1}$, in which case kernel is the const. fns

③ The image $\hat{\mathcal{F}}_\mu = \{ \psi$ on $F^\times$ sm $|$

$\psi(x) = 0$ if x big.

$$\psi(x) = \begin{cases} a\mu(x) + b & \text{if } \mu \neq \mathbb{1} \text{ or } |\ |^{-1} \\ a\, v(x) + b & \text{if } \mu = \mathbb{1} \\ b & \text{if } \mu = |\ |^{-1}. \end{cases}$$

(valuation)

for some a, b const. if x near 0. $\}$

Pf: $\mu(x)|x|\Phi(x)$ const for x large.

$\Rightarrow \mathcal{F}_\mu = \mathcal{S}(F) \oplus \boxed{\mathbb{C}\,\Phi_\mu}$

$$\Phi_\mu := \begin{cases} \mu^{-1}(x)|x|^{-1} & |x| \geq 1 \\ 0 & |x| < 1 \end{cases}$$

$$\hat{\Phi}_\mu(x) = \sum_{n \leq 0} \int_{\varpi^n \mathcal{O}_F^\times} \mu^{-1}(y)\, \overline{\tau_F(xy)}\, \frac{dy}{|y|}$$

Step 1

Compute this to get convergence + behaviour near 0.

A) μ is ramified

Let $\wp^f$ be the conductor of μ $\mu|_{U_f} = \mathbb{1}$. smallest such f

$\wp^d$ — — — τ_F $\tau_F|_{\varpi^d \mathcal{O}_F} = \mathbb{1}$. ~~~ d.

Codement's notes used $-d$.

$$\circledast \int_{\varpi^n \mathcal{O}_F^\times} \mu^{-1}(y)\, \overline{\tau_F(xy)}\, \frac{dy}{|y|} = \int_{\mathcal{O}_F^\times} \mu^{-1}(\varpi^n u)\, \overline{\tau_F(\varpi^n x u)}\, du$$

$$= \sum_{a \in \mathcal{O}_F^\times / U_f} \int_{aU_f} \mu^{-1}(\varpi^n u)\, \overline{\tau_F(\varpi^n x u)}\, du$$

$$= \sum_a \mu^{-1}(\varpi^n) \int_{a\mathcal{U}_f} \mu^{-1}(u)\, \overline{\tau_F(\varpi^n x u)}\, du$$

$$= \sum_a \mu^{-1}(\varpi^n a) \int_{\mathcal{U}_f} \overline{\tau_F(\varpi^n x a u)}\, du$$

$$= \sum_a \mu^{-1}(\varpi^n a)\, \overline{\tau_F(\varpi^n x a)} \int_{\varpi^f \mathcal{O}_F} \overline{\tau_F(\varpi^n x a y)}\, dy$$

$$= \sum_a \mu^{-1}(\varpi^n a)\, \overline{\tau_F(\varpi^n x a)}\, q^{-f} \int_{\mathcal{O}_F} \overline{\tau_F(\boxed{\varpi^{n+f} x a} y)}\, dy$$

$\in \varpi^d \mathcal{O}_F \quad = 1$

integral is nonvan $\Rightarrow$ $x \in \varpi^{d-n-f}\mathcal{O}_F$.

outside the Gauß sum is nonvan. $\Rightarrow$ $x \in \varpi^{d-n-f}\mathcal{O}_F^\times$.

$\circledast \neq 0 \iff x \in \varpi^{d-n-f}\mathcal{O}_F^\times$.

$\rightsquigarrow$ convergence.

$$\hat{\Phi}_\mu(x) = \int_{\varpi^n \mathcal{O}_F^\times} \mu^{-1}(y)\, \overline{\tau_F(xy)}\, \frac{dy}{|y|} = \int_{\varpi^{d-f}\mathcal{O}_F^\times} \mu^{-1}(x^{-1}y)\, \overline{\tau_F(y)}\, \frac{dy}{|y|}$$

$n = d - f - v(x)$ $\qquad = \mu(x) * \text{const.}$

B) μ is unramified. $\mu = |\ |^s$

$$\int_{\varpi^n \mathcal{O}_F^\times} \overline{\tau_F(xy)}\, |y|^{-s} \frac{dy}{|y|} = q^{ns} \int_{\mathcal{O}_F^\times} \overline{\tau_F(\varpi^n x u)}\, du.$$

$$= q^{ns} \left(\int_{\mathcal{O}_F} - \int_{\varpi \mathcal{O}_F} \right) = q^{ns} \left(h(\varpi^n x) - q^{-1} h(\varpi^{n+1} x) \right)$$

$$h(x) := \int_{\mathcal{O}_F} \overline{\tau_F(\ \ x u)}\, du = \begin{cases} 1 & v(x) \geq d \\ 0 & v(x) < d \end{cases}$$

$$\hat{\Phi}_\mu(x) = \sum_{n \leq 0} q^{ns} \left(h(\varpi^n x) - q^{-1} h(\varpi^{n+1} x) \right)$$ For fixed x, finite sum.

$$= \sum_{d - v(x) \leq n \leq 0} q^{ns} - \sum_{d - v(x) - 1 \leq n \leq 0} q^{ns} q^{-1}$$ $\rightsquigarrow$ get convergence.

① $q^s \neq 1$ i.e. $\mu \neq \mathbb{1}$.

$$= \frac{1 - q^{-(v(x) - d + 1)s}}{1 - q^{-s}} - \frac{1 - q^{-(v(x) - d + 2)s}}{1 - q^{-s}} \cdot q^{-1} = a|x|^s + b$$

$\sim |x|^s$

$$a = \frac{q^{(d-1)s} - q^{(d-2)s} q^{-1}}{1 - q^{-s}} \begin{cases} \neq 0 & \text{if } s \neq -1 \\ = 0 & \text{if } s = -1 \end{cases}$$

② $q^s = 1$ i.e., $\mu = \mathbb{1}$.

$$\hat{\Phi}_\mu(x) = v(x) - d + 1 + q^{-1}(v(x) - d + 2) \sim v(x)$$

Step 2

Kernel of $\Phi \longmapsto \hat{\Phi}$. For $\forall f \in \mathcal{S}(F^\times)$ extend f to F

$f(0) = 0$.

$\rightsquigarrow f \in \mathcal{S}(F)$.

$$\int_{F^\times} f(x)\, \hat{\Phi}(x)\, dx$$

$$= \int_{F^\times} \underbrace{f(x)}_{} \sum_{n \in \mathbb{Z}} \int_{\varpi^n \mathcal{O}_F^\times} \Phi(y)\, \overline{\tau_F(xy)}\, dy$$

cpt supp $\rightsquigarrow$ actually fin sum here

$$= \sum_{n \in \mathbb{Z}} \int_{F^\times} \int_{\varpi^n \mathcal{O}_F^\times} f(x)\, \overline{\tau_F(xy)}\, \Phi(y)\, dy$$

$$= \sum_{n \in \mathbb{Z}} \int_{\varpi^n \mathcal{O}_F^\times} \hat{f}(y)\, \Phi(y)\, dy$$

$\mathcal{S}(F)$

$$= \int_F \hat{f}(y)\, \Phi(y)\, dy. \qquad = \langle \tilde{\Phi}, f \rangle$$

$\hat{\Phi} = 0 \iff \tilde{\Phi}$ proportional to Dirac distribution (b/c it vanishes on all of $\mathcal{S}(F^\times)$)

$\iff \Phi$ const function. $\quad \langle \tilde{\Phi}, f \rangle = c \hat{f}(0) = c \int_F f(y)\, dy.$

$\iff \mu = |\ |^{-1}$.

$\square$